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Topology of the isospectral real manifolds associated with the generalized Toda lattices on semisimple Lie algebras

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Abstract

This paper concerns the topology of isospectral real manifolds of certain Jacobi elements associated with real split semisimple Lie algebras. The manifolds are related to the compactified level sets of the generalized (non-periodic) Toda lattice equations defined on the semisimple Lie algebras. We then give a cellular decomposition and the associated chain complex of the manifold by introducing coloured Dynkin diagrams which parametrize the cells in the decomposition. We also discuss the Morse chain complex of the manifold.

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1. The generalized Toda lattice equations

Let $\mathfrak{g}$ denote a real split semisimple Lie algebra of rank l. We fix a split Cartan subalgebra $\mathfrak{h}$ with root system Δ, real root vectors e_{α_i} associated with simple roots $\{\alpha_i : i = 1, \ldots, l\} = \Pi$. We also denote by $\{h_{\alpha_i}, e_{\pm\alpha_i}\}$ the Cartan–Chevalley basis of $\mathfrak{g}$ which satisfies the relations

$$[h_{\alpha_i}, h_{\alpha_j}] = 0 \qquad [h_{\alpha_i}, e_{\pm\alpha_j}] = \pm C_{j,i} e_{\pm\alpha_j} \qquad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{i,j} h_{\alpha_j} \tag{1.1}$$

where the $l \times l$ matrix $(C_{i,j})$ is the Cartan matrix corresponding to $\mathfrak{g}$, and $C_{i,j} = \alpha_i(h_{\alpha_j}) = \langle \alpha_i, h_{\alpha_j} \rangle$.

Then the generalized Toda lattice equation related to the real split semisimple Lie algebra is defined by the following system of second-order differential equations for the real variables $\{f_j(t) : j = 1, \ldots, l\}$:

$$\frac{\mathrm{d}^2 f_i}{\mathrm{d}t^2} = \epsilon_i \exp\left(-\langle \alpha_i, f \rangle\right) \tag{1.2}$$

where $f = \sum_{j=1}^{l} f_j(t) h_{\alpha_j} \in \mathfrak{h}$ and $\epsilon_i \in \{\pm 1\}$.

Remark 1.1. The case with $\mathfrak{g} = \mathfrak{sl}(l+1, \mathbb{R})$ corresponds to the indefinite Toda lattice introduced in [7]. The main feature of the indefinite Toda equation having at least one of ϵ_i being -1 is that the solution blows up to infinity in finite time [7]. Having introduced the signs, the group corresponding to the Toda lattice is a real split Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$. For example, in the case of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, if n is odd, $\tilde{G} = SL(n, \mathbb{R})$, and if n is even, $\tilde{G} = Ad(SL(n, \mathbb{R})^{\pm})$.

Remark 1.2. If we consider the complex Toda equation, ϵ_i in (1.2) can be absorbed in $f_i \in \mathbb{C}$, so that the present study deals with the disconnected Cartan subgroup, where the generalized Toda lattice defines a flow in each connected component.

Remark 1.3. The original Toda lattice in [12] is obtained as the case with all $\epsilon_i = 1$ where the position of the ith particle is given by $q_i = f_i - f_{i+1}$ for $i = 1, \ldots, l$,

$$\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} = \exp(q_{i-1} - q_i) - \exp(q_i - q_{i+1}) \tag{1.3}$$

where $f_{l+1} = 0$ and $f_0 = f_{l+2} = -\infty$ indicating $q_0 = -\infty$ and $q_{l+1} = \infty$.

1.1. Lax formulation: isospectral manifold $Z(\gamma)_{\mathbb{R}}$

The system (1.2) can be written in a Lax equation which describes an isospectral deformation of a Jacobi element of $\mathfrak{g}$ [4]. Define the set of real functions $\{(a_i(t), b_i(t)) : i = 1, \ldots, l\}$,

$$a_i(t) = \frac{\mathrm{d}}{\mathrm{d}t} f_i(t) \qquad b_i(t) = \epsilon_i \exp\left(-\langle \alpha_i, f \rangle\right). \tag{1.4}$$

Then the Toda equation (1.2) can be written in the Lax form [4, 9],

$$\frac{\mathrm{d}X}{\mathrm{d}t} = [P, X] \tag{1.5}$$

where the Lax pair (X, P) are defined by

$$\begin{aligned} X(t) &= \sum_{i=1}^{l} a_i(t) h_{\alpha_i} + \sum_{i=1}^{l} \left(b_i(t) e_{-\alpha_i} + e_{\alpha_i} \right) \\ P(t) &= -\sum_{i=1}^{l} b_i(t) e_{-\alpha_i}. \end{aligned} \tag{1.6}$$

The Lax form (1.5) represents an isospectral deformation of the Jacobi element X.

We denote the disconnected manifold given by the set of the elements in the form X of $\mathfrak{g}$,

$$Z_{\mathbb{R}} = \left\{ X = x + \sum_{i=1}^{l} (e_{\alpha_i} + b_i e_{-\alpha_i}) \in \mathfrak{g} : x \in \mathfrak{h}, b_i \in \mathbb{R}^* \right\} = \bigcup_{\epsilon \in \mathcal{E}} Z_\epsilon \tag{1.7}$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, and the connected component Z_ϵ is given by

$$Z_\epsilon = \{X \in Z_{\mathbb{R}} : \epsilon = (\epsilon_1, \ldots, \epsilon_l) \in \mathcal{E}, \operatorname{sign}(b_i) = \epsilon_i\}. \tag{1.8}$$

Here $\mathcal{E}$ is the set of all the signs ϵ, so the set $Z_{\mathbb{R}}$ is the disjoint union of the 2^l connected components.

A real isospectral leaf in $Z_{\mathbb{R}}$ is defined by the level sets of the Chevalley invariants, denoted as $(I_1, \ldots, I_l)$, which are the polynomials of the variables (a_i, b_i). The invariants then define a differentiable map,

$$\begin{aligned} \mathcal{I} : Z_{\mathbb{R}} &\longrightarrow \mathbb{R}^l \\ X &\longmapsto \gamma = (I_1, \ldots, I_l). \end{aligned} \tag{1.9}$$

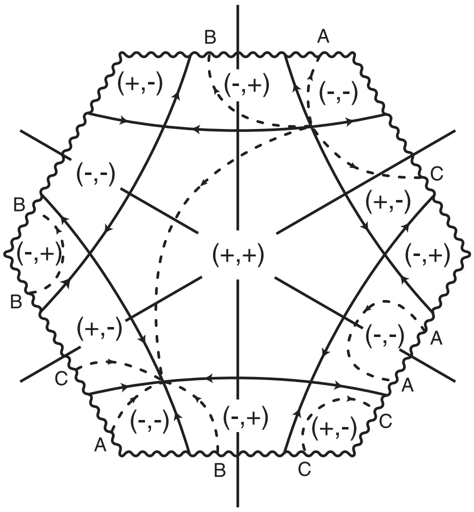

Figure 1. The isospectral manifold $\hat{Z}(\gamma)_{\mathbb{R}}$ for $\mathfrak{sl}(3, \mathbb{R})$.

The real isospectral leaf $Z(\gamma)_{\mathbb{R}}$ is then given by

$$Z(\gamma)_{\mathbb{R}} = \mathcal{I}^{-1}(\gamma) \bigcap Z_{\mathbb{R}}. \tag{1.10}$$

Our main purpose in this paper is to give a detailed structure of the compactified manifold $\hat{Z}(\gamma)_{\mathbb{R}}$ from the viewpoint of the Lie group theory.

Remark 1.4. The Chevalley invariants provide l-involutive integrals for the generalized Toda lattice equation, so that this proves the integrability of the equation in the Liouville–Arnold sense.

Remark 1.5. The construction of the compactified manifold $\hat{Z}(\gamma)_{\mathbb{R}}$ for the case of $\mathfrak{sl}(l+1, \mathbb{R})$ was given in [8] based on the explicit solution structure in terms of the τ-functions, which provide a local coordinate system for the manifold. By tracing the solution orbit of the indefinite Toda equation, the disconnected components in $Z(\gamma)_{\mathbb{R}}$ are all glued together to make a smooth compact manifold. The result is possibly well explained by figure 1 for the case of $A_2 \cong \mathfrak{sl}(3, \mathbb{R})$. In the figure, the Toda orbits are shown as the dotted curves, and each region labelled by the same signs in (ϵ_1, ϵ_2) with $\epsilon_i \in \{\pm\}$ are glued together through the boundary (the wavy curves) of the hexagon. At a point of the boundary the Toda orbit blows up in finite time, but the orbit can be uniquely traced to the one in the next region (marked by the same letter A, B or C). Also the flows on the full lines show the solutions of the subsystems (i.e. either $b_1 = 0$ or $b_2 = 0$). Then the compactified manifold $\hat{Z}(\gamma)_{\mathbb{R}}$ by adding the blow-up points (the wavy lines) and the flows of the subsystems (the solid lines) to $Z(\gamma)_{\mathbb{R}}$ in this case is shown to be isomorphic to the connected sum of two Klein bottles, that is, the integral homology $H_k(\hat{Z}(\gamma)_{\mathbb{R}}, \mathbb{Z})$ is given by $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}^3 \oplus \mathbb{Z}_2$, and $H_2 = 0$. In the case of $\mathfrak{sl}(n, \mathbb{R})$ for $n \geqslant 3$, $\hat{Z}(\gamma)_{\mathbb{R}}$ is shown to be non-orientable and the symmetry group is the semi-direct product of $(\mathbb{Z}_2)^{n-1}$ and the Weyl group $W = S_n$, the permutation group. One should compare this with the result of Tomei [13] where the compact manifolds are associated with the definite (original) Toda lattice equation and the compactification is done by adding only the subsystems. (Also see [3] for some topological aspects of the manifolds which are identified as permutohedrons.)

1.2. Leznov–Saveliev formulation: Cartan subgroup $H_{\mathbb{R}}$

In the zero-curvature formulation in [10], the generalized Toda lattice equation (1.2) with the sign $\epsilon = (\epsilon_1, \ldots, \epsilon_l)$ can be expressed as an orbit on the connected component H_ϵ of the Cartan subgroup $H_{\mathbb{R}}$,

$$H_{\mathbb{R}} = \bigcup_{\epsilon \in \mathcal{E}} H_\epsilon \tag{1.11}$$

where $H_{(1,\ldots,1)} := H = \exp \mathfrak{h}$, the connected component with the identity. Thus the set $H_{\mathbb{R}}$ consists of 2^l connected components. Let g_ϵ be an element of H_ϵ given by

$$g_\epsilon = h_\epsilon \exp f \tag{1.12}$$

which can also be considered as a map from $Z_{\mathbb{R}}$ to $H_{\mathbb{R}}$. Here the element $h_\epsilon \in H_\epsilon$ satisfies $\chi_{\alpha_i}(h_\epsilon) = \epsilon_i$ with the group character χ_ϕ determined by a root $\phi \in \Delta$, and each connected component of $H_{\mathbb{R}}$ can be written as $H_\epsilon = h_\epsilon H$. Then the Toda lattice (1.2) is written as an evolution of $g_\epsilon(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} g_\epsilon^{-1} \frac{\mathrm{d}}{\mathrm{d}t} g_\epsilon = \left[g_\epsilon^{-1} e_+ g_\epsilon, e_- \right] \tag{1.13}$$

where $e_\pm$ are fixed elements in the simple root spaces $\mathfrak{g}_{\pm\Pi}$ so that all the elements in $\mathfrak{g}_{\pm\Pi}$ can be generated by $e_\pm$, i.e. $\mathfrak{g}_{\pm\Pi} = \{Ad_h(e_\pm) : h \in H\}$. In particular, we take

$$e_\pm = \sum_{i=1}^{n} e_{\pm\alpha_i}. \tag{1.14}$$

With the group character χ_{α_i}, the solution $b_i(t)$ of the Toda lattice is given by

$$b_i(t) = \left[\chi_{\alpha_i}(g_\epsilon)\right]^{-1} = \chi_{-\alpha_i}(g_\epsilon). \tag{1.15}$$

Remark 1.6. With the fundamental weights ω_i defined as $\langle \omega_i, h_{\alpha_j} \rangle = \delta_{ij}$, i.e. $\alpha_i = \sum_{j=1}^{l} C_{i,j}\omega_j$, we can write the solution

$$b_i = \prod_{j=1}^{l} \left[\chi_{\omega_j}(g_\epsilon)\right]^{-C_{i,j}} \tag{1.16}$$

which is the well known τ-function representation of the solution with $\tau_i(t) := \chi_{\omega_i}(g_\epsilon)$.

Remark 1.7. In the compactification of the disconnected Cartan subgroup $H_{\mathbb{R}}$, we need to add pieces corresponding to the blow-ups ($|b_i| = \infty$) and the subsystems ($b_i = 0$). The subsystems are determined by the subset $A = \{\alpha_i \in \Pi : b_i = 0\}$, and the corresponding Cartan subgroup, denoted by $H_{\mathbb{R}}^A$, may be defined as

$$H_{\mathbb{R}}^A = \bigcup_{\epsilon \in \mathcal{E}^A} h_\epsilon H^A \tag{1.17}$$

where the set $\mathcal{E}^A \subset \mathcal{E}$ and the Cartan subgroup H^A are defined by

$$\mathcal{E}^A = \{(\epsilon_1, \ldots, \epsilon_l) \in \mathcal{E} : \epsilon_i = 1 \text{ if } \alpha_i \in A\} \tag{1.18}$$

$$H^A = \exp \mathfrak{h}^A \qquad \text{with} \quad \mathfrak{h}^A = \mathrm{Span}_{\mathbb{R}}\{h_{\alpha_i} \in \mathfrak{h} : \alpha_i \notin A\}. \tag{1.19}$$

Then the subsystems are also expressed as the same form of (1.13) with $g_\epsilon^A \in H_\epsilon^A$,

$$g_\epsilon^A = h_\epsilon \exp\left(\sum_{\alpha_i \notin A} f_i(t) h_{\alpha_i}\right). \tag{1.20}$$

The corresponding Lax pair (X^A, P^A) is given by

$$\begin{aligned} X^A &= \sum_{\alpha_i \in \Pi} a_i h_{\alpha_i} + \sum_{\alpha_i \notin A} b_i e_{-\alpha_i} + e_+ \\ P^A &= -\sum_{\alpha_i \notin A} b_i e_{-\alpha_i} \end{aligned} \tag{1.21}$$

which is just the Lax pair (1.6) with $b_i = 0$ for $\alpha_i \in A$. Note here that $a_i(t) =$ constant if $b_i = 0$. We also consider that the dimension of the manifold $H^A_{\mathbb{R}}$ is $l - |A|$, the number of parameters f_i.

Remark 1.8. The compactification of the isospectral manifold $Z(\gamma)_{\mathbb{R}}$ for a fixed $\gamma \in \mathbb{R}^l$ can be obtained by sending it to the flag manifold $\tilde{G}/B_+$ with the Borel subgroup B_+ of $\tilde{G}$ [9], so that the compactified manifold $\hat{Z}(\gamma)_{\mathbb{R}}$ is a toric variety $\overline{H_{\mathbb{R}} x B_+}$ with a generic element $x \in \tilde{G}$.

Then we can show:

Theorem 1.1. *The isospectral manifold $\hat{Z}(\gamma)_{\mathbb{R}}$ is a smooth compact manifold diffeomorphic to $\hat{H}_{\mathbb{R}}$.*

The complex version of this theorem is given in [5], and the proof of the present case is essentially given in the same manner (the detail of the proof is given in [2]).

In the following two sections, we will describe the structure of $\hat{H}_{\mathbb{R}}$ using the Weyl action on the manifold. This is a brief summary of a preprint [2], and the proofs of the results (proposition 2.1, theorems 3.1 and 3.2) can be found therein. Then in section 4, we will present the Morse theory to compute the integral homology of the manifold $\hat{H}_{\mathbb{R}}$.

2. The structure of $\hat{H}_{\mathbb{R}}$ as the union of the subsystems

As was shown in the previous section, the set $H_{\mathbb{R}}$ can be parametrized by the group characters χ_{α_i}, that is, $H_{\mathbb{R}} = \cup_{\epsilon \in \mathcal{E}} H_\epsilon$ with $H_\epsilon = h_\epsilon H$,

$$H_\epsilon = \left\{ h \in H_{\mathbb{R}} : \mathrm{sign}(\chi_{\alpha_i}(h)) = \chi_{\alpha_i}(h_\epsilon) = \epsilon_i \text{ for } i = 1, \ldots, l \right\}. \tag{2.1}$$

Note that each H_ϵ is diffeomorphic to $\mathbb{R}^l$.

Since the Weyl group acts on $H_{\mathbb{R}}$, one can partition $H_{\mathbb{R}}$ into the $|W|$ convex cones of the Weyl chambers. We denote the cone in the antidominant chamber as

$$H^-_{\mathbb{R}} = \bigcup_{\epsilon \in \mathcal{E}} H^-_\epsilon \tag{2.2}$$

where the connected component H^-_ϵ is defined by

$$H^-_\epsilon = \left\{ h \in H_{\mathbb{R}} : |\chi_{-\alpha_i}(h)| \leqslant 1, \mathrm{sign}(\chi_{\alpha_i}(h)) = \epsilon_i \right\}. \tag{2.3}$$

The boundaries of the chamber H^-_ϵ corresponding to $\chi_{\alpha_i}(h) = 1$ and $\chi_{\alpha_i}(h) = -1$ are called the positive and negative α_i-walls, and, in particular, the positive α_i-wall gives the hyperplane of the Weyl reflection with respect to the root α_i. Then the connected component H_ϵ of $H_{\mathbb{R}}$ is expressed as the union of W-translations of H^-_ϵ, i.e.

$$H_\epsilon = \bigcup_{w \in W} w\left(H^-_{\epsilon(w)}\right). \tag{2.4}$$

Here the W-action on H^-_ϵ is obtained through the action on the group characters,

$$s_{\alpha_i}(\chi_{\alpha_j}) = \chi_{s_{\alpha_i}\alpha_j} = \chi_{\alpha_j} \chi_{\alpha_i}^{-C_{j,i}} \tag{2.5}$$

from which we also have the W-action on the set $\mathcal{E}$ as $s_{\alpha_i} : \epsilon_j \mapsto \epsilon'_j$ with $\epsilon_j = \chi_{\alpha_j}(h_\epsilon)$,

$$\epsilon'_j = \epsilon_j \epsilon_i^{-C j,i} \tag{2.6}$$

where we have used the Weyl reflection, $s_{\alpha_i}\alpha_j = \alpha_j - C_{j,i}\alpha_i$. Thus the element h_ϵ is W-translated to $h_{\epsilon'}$ with the sign $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_l)$ given by (2.6), which we denote by $\epsilon' = \epsilon(w)$ with $w = s_{\alpha_i}$ Weyl reflection. Thus, with the decomposition (2.4), we can consider only the antidominant chamber H_ϵ^-, and obtain the whole $H_\mathbb{R}$ by the W-translates. This is also true for the compactified manifold $\hat{H}_\mathbb{R}$.

Let us first make the closure of H_ϵ^- by adding the pieces corresponding to the subsystems having the lower dimensions $l - |A|$ where $A \subset \Pi$ determines the subsystem (see remark 1.7). We let

$$H_\epsilon^{A,-} = \left\{ h \in H_\epsilon^A : |\chi_{-\alpha_i}(h)| \leqslant 1, \epsilon = (\epsilon_{i_1}, \ldots, \epsilon_{i_m}) \text{ for } \alpha_{i_j} \notin A \right\}. \tag{2.7}$$

Then the closure of the set H_ϵ^- can be obtained by

$$\overline{H_\epsilon^-} = \bigcup_{A \subset \Pi} H_\epsilon^{A,-} \tag{2.8}$$

and the compactified manifold $\hat{H}_\mathbb{R}$ is given by the W-translates of (2.8), i.e.

$$\hat{H}_\mathbb{R} = \bigcup_{\epsilon \in \mathcal{E}} \bigcup_{w \in W} w\big(\overline{H_{\epsilon(w)}^-}\big). \tag{2.9}$$

We summarize the result as:

Proposition 2.1. *The closed set* $\overline{H_\epsilon^-}$ *is isomorphic to the box* $\{(t_1, \ldots, t_l) \in \mathbb{R}^l : -1 \leqslant t_j \leqslant 1\}$, *and the manifold* $\hat{H}_\mathbb{R}$ *is compact and has an action of the Weyl group* W.

3. Topology of $\hat{H}_\mathbb{R}$

3.1. Coloured Dynkin diagrams

Here we give a cellular decomposition and construct the associated chain complex of the compactified manifold $\hat{H}_\mathbb{R}$. We first introduce the set of coloured Dynkin diagrams to parametrize the cells in the decomposition. A coloured Dynkin diagram is simply a Dynkin diagram in which some of the vertices have been coloured either red (R) or blue (B). For example, in the case of $A_2 \cong \mathfrak{sl}(3, \mathbb{R})$, we have $\circ_R - \circ$, $\circ_B - \circ_R$, etc. Thus a coloured Dynkin diagram D corresponds to a pair (S, η) with $S \subset \Pi$ and $\eta : S \to \{\pm 1\}$, where $\eta(\alpha_i) = -1$ if α_i is coloured R, and $\eta(\alpha_i) = 1$ if α_i is coloured B. We denote the set of coloured Dynkin diagrams as

$$\mathbb{D}(S) = \{D = (S, \eta) : S \subset \Pi, \eta(\alpha) \in \{\pm 1\} \text{ for } \alpha \in S\}. \tag{3.1}$$

Let W_S be the group generated by the simple reflections corresponding to the roots in S. We then define the W_S-action on the set $\mathbb{D}(S)$ as follows: for any $\alpha_i \in S$, $s_{\alpha_i} D = D'$ is a new coloured Dynkin diagram having the colours corresponding to the sign change $\epsilon'_j = \epsilon_j \epsilon_i^{-C_{j,i}}$ in (2.6) with the identification that R if the sign is -1, and B if it is +1. For example, in the case of A_2, we have $s_{\alpha_1}(\circ_R - \circ_B) = \circ_R - \circ_R$, $s_{\alpha_1}(\circ_B - \circ_R) = \circ_B - \circ_R$.

The W_S-action induces the W-translates on the set $\mathbb{D}(S)$ as $W \times_{W_S} \mathbb{D}(S)$. The elements of this set are given by pairs (w, D), and among the elements we have an equivalence relation

$\sim$, that is, $(wx, D) \sim (w, xD)$ for any $x \in W_S$. The equivalent relation gives a bijective correspondence between $W \times_{W_S} \mathbb{D}(S)$ and $\mathbb{D}(S) \times W/W_S$. We then define the set $\mathbb{D}^k$ as

$$\mathbb{D}^k := \{(D, [w]_{\Pi-S}) : D \in \mathbb{D}(S), [w]_{\Pi-S} \in W/W_S, |S| = k\} \tag{3.2}$$

which parametrizes all the connected components of the Cartan subgroups of the form $H_{\mathbb{R}}^{\Pi-S}$ corresponding to the subsystems defined in remark 1.7. In this parametrization, $\mathbb{D}^k$ corresponds explicitly to the dual of the set $H_{\mathbb{R}}^{\Pi-S}$, so that the parametrized cell has the codimension k, and $\dim H_{\mathbb{R}}^{\Pi-S} = k$. Thus all the cells in $\hat{H}_{\mathbb{R}}$ can be parametrized by the sets $\mathbb{D}^k$, and we have:

Theorem 3.1. *The collection of the sets $\mathbb{D}^k$ defined (3.2) gives a cell decomposition of the compact manifold $\hat{H}_{\mathbb{R}}$.*

Remark 3.1. There is a more convenient cell decomposition of $\hat{H}_{\mathbb{R}}$ for the purpose of calculating homology explicitly. The only change is that the l-dimensional cell becomes the union of all the l-cells together with all the (internal) boundaries corresponding to coloured Dynkin diagrams where all the coloured vertices are coloured B. This is the set,

$$\hat{H}_{\mathbb{R}} \setminus \bigcup_{\substack{S\subset\Pi, w\in W \\ \eta(\alpha_i)=-1 \text{ for some } \alpha_i\in S}} (S, \eta, [w]_{\Pi-S}).$$

This set can be seen to be homeomorphic to $\mathbb{R}^l$. With this cell decomposition there is exactly one l-cell, and the other lower-dimensional cells correspond to coloured Dynkin diagrams D, in which at least one vertex of D has been coloured R.

Example 3.1. *In the case of A_2, we have:*

(a) *For $k = 2$, i.e. $S = \Pi$, we have four vertices (0-cell) parametrized by the elements of $\mathbb{D}^2$, which correspond to the four connected components of $H_{\mathbb{R}}$ as dual cells,*

$$(\circ_B - \circ_B, [e]) \quad (\circ_B - \circ_R, [e]) \quad (\circ_R - \circ_B, [e]) \quad (\circ_R - \circ_R, [e])$$

where $[e] = [e]_\emptyset = [w]_\emptyset$ for any $w \in W$.

(b) *For $k = 1$, if $S = \{\alpha_1\}$, we have six 1-cells parametrized by*

$$(\circ_B - \circ, [w]_{\{\alpha_2\}}) \quad (\circ_R - \circ, [w]_{\{\alpha_2\}})$$

where $W/W_S = \{e, s_{\alpha_2}, s_{\alpha_1}s_{\alpha_2}\}$, and if $S = \{\alpha_2\}$, we have also six 1-cells,

$$(\circ - \circ_B, [w]_{\{\alpha_1\}}) \quad (\circ - \circ_R, [w]_{\{\alpha_1\}})$$

where $W/W_S = \{e, s_{\alpha_1}, s_{\alpha_2}s_{\alpha_1}\}$. Those coloured Dynkin diagrams with $w = e$ correspond to the four walls (2-positive and 2-negative walls) of the antidominant chamber $H_{\mathbb{R}}^-$, which is isomorphic to a square.

(c) *For $k = 0$, i.e. $S = \emptyset$, we have six 2-cells of the convex cones corresponding to the Weyl chambers parametrized by*

$$(\circ - \circ, w)$$

for $w \in W$. Those are dual to the 6-vertices corresponding to $H_{\mathbb{R}}^{\Pi}$. As mentioned in remark 3.1, we have a simpler cell decomposition. Namely, the union of all coloured Dynkin diagrams having no R-coloured vertices forms the unique 2-cell which is the set of internal points of the hexagon homeomorphic to $\mathbb{R}^l$, and all other cells consist of the boundary of the hexagon.

Figure 2 illustrates the example. One should note that full parametrization of the cells is obtained by the W-translates of $(D, [e]_{\Pi-S})$ corresponding to the subsystems in the antidominant chamber $H_{\mathbb{R}}^{\Pi-S,-}$ for all the choices of $S \subset \Pi$.

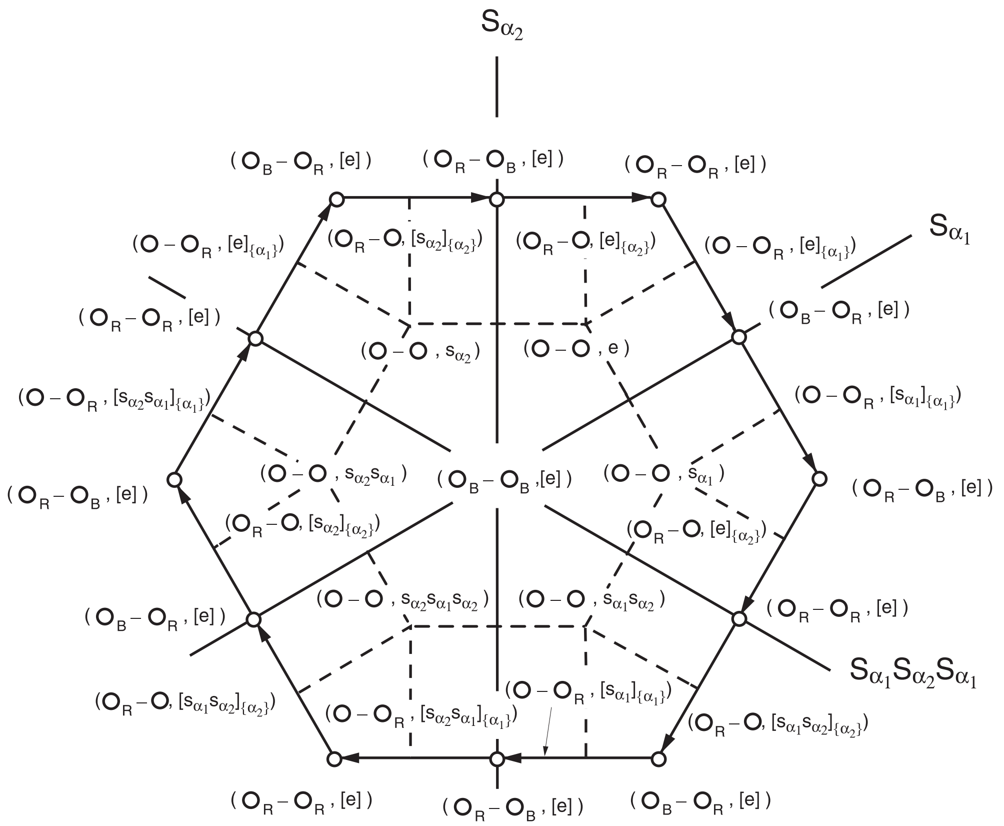

Figure 2. The manifold $\hat{H}_{\mathbb{R}}$ parametrized by coloured Dynkin diagrams for $\mathfrak{sl}(3,\mathbb{R})$.

3.2. Boundary maps and chain complex

With these parametrizations of the cells, we now define the boundary maps on the coloured Dynkin diagrams. Let (j, c) be a pair of integers with $j = 1, \ldots, m$ and $c = 1, 2$. Then we define the (j, c)-boundary, denoted as $\partial_{j,c}D$, of a coloured Dynkin diagram D with the set of uncoloured vertices $\{\alpha_{i_j} : 1 \leqslant i_1 \ldots, i_m \leqslant l\}$ as a new coloured Dynkin diagram by colouring the i_jth vertex with R if $c = 1$ and with B if $c = 2$. Recall that a coloured Dynkin diagram D corresponds to a pair (S, η) with $S \subset \Pi$ and $\eta : S \to \{\pm 1\}$. Thus the boundary operator $\partial_{j,c}$ determines a new pair $(S \cup \{\alpha_{i_j}\}, \eta')$ where η' is an extension of the η on $S \cup \{\alpha_{i_j}\}$ with $\eta'(\alpha_{i_j}) = (-1)^c$. Giving an orientation on the boundary, we define the map $\tilde{\partial}_{j,c} := (-1)^{j+c+1}\partial_{j,c}$ which gives a map on the $\mathbb{Z}$-modules,

$$\tilde{\partial}_{j,c} : \mathbb{Z}[\mathbb{D}(S)] \quad \longrightarrow \quad \mathbb{Z}[\mathbb{D}(S \cup \{\alpha_{i_j}\})]. \tag{3.3}$$

Let us now consider the $\mathbb{Z}$-modules of the full set of coloured Dynkin diagrams, $\mathbb{D}^k = \mathbb{D}(S) \times W/W_S|_{|S|=k}$. We denote the module as

$$\mathcal{M}(S) = \mathbb{D}[W] \otimes_{\mathbb{Z}[W_S]} \mathbb{Z}[\mathbb{D}(S)] \tag{3.4}$$

so that the k-chain is given by the direct sum of all these modules over all sets with $|S| = l - k$,

$$\mathcal{M}_k = \bigoplus_{|S|=l-k} \mathcal{M}(S). \tag{3.5}$$

The boundary map $\partial_k : \mathcal{M}_k \to \mathcal{M}_{k-1}$ can be defined by

$$\partial_k \left(D, [w]_{\Pi-S}\right) = \sum_{\substack{1\leqslant j\leqslant k\\ c=1,2}} \left(\tilde{\partial}_{j,c} D, [w]_{\Pi-\{S\cup\{\alpha_{i_j}\}\}}\right) \tag{3.6}$$

where $\Pi - S = \{\alpha_{i_j} : 1 \leqslant j \leqslant k\}$. The condition for the boundary map, $\partial_k \circ \partial_{k+1} = 0$, is then easily verified, and we have:

Theorem 3.2. *The map ∂_k of (3.6) defines a chain complex $\mathcal{M}_*$,*

$$0 \longrightarrow \mathcal{M}_l \xrightarrow{\partial_l} \mathcal{M}_{l-1} \xrightarrow{\partial_{l-1}} \cdots \xrightarrow{\partial_2} \mathcal{M}_1 \xrightarrow{\partial_1} \mathcal{M}_0 \longrightarrow 0 \tag{3.7}$$

where $\mathcal{M}_k$ is defined by (3.5).

Since we have an explicit representation of the k-chains as (3.5) with (3.4), the integral homology $H_k(\hat{H}_{\mathbb{R}}; \mathbb{Z}) = \mathrm{Ker}\ \partial_k / \mathrm{Im}\ \partial_{k+1}$ can also be computed. However, an explicit formula may be too complicated.

4. Morse theory and homology

4.1. Morse theory

The generalized Toda equation can be expressed as a gradient flow on the adjoint orbit of $\mathfrak{g}$ [1]. Here we consider a Morse decomposition of the manifold $\hat{H}_{\mathbb{R}}$ based on the gradient structure of the Toda flow. Each critical point of the Toda vector field can be parametrized by a unique element of the Weyl group W. Then we define the *unstable* and *stable* Weyl subgroups for $a \in W$ as

$$\begin{aligned} W^u(a) &= W_{\Pi^u_a} \qquad && \Pi^u_a := \{\alpha_i \in \Pi : \ell(as_{\alpha_i}) > \ell(a)\} \\ W^s(a) &= W_{\Pi^s_a} \qquad && \Pi^s_a := \{\alpha_i \in \Pi : \ell(as_{\alpha_i}) < \ell(a)\} \end{aligned} \tag{4.1}$$

where $\ell(a)$ denotes the length of a. We will use the same notation for the unstable and stable manifolds generated by the Toda vector field corresponding to the critical point a. Thus, depending on the context, $W^u(a)$, $W^s(a)$ denotes either a subgroup of W or a submanifold of $\hat{H}_{\mathbb{R}}$. We also introduce labels in the Dynkin diagram to characterize the critical point by assigning '0' in the ith place in the diagram if $s_{\alpha_i} \in W^s(a)$, and '$*$' if $s_{\alpha_i} \in W^u(a)$. For example, in the case of $\mathfrak{g} = \mathfrak{sl}(6 : \mathbb{R})$, the element $[2143] \in W$ is labelled as $(0 * 0 * *)$, where $[2143] := s_{\alpha_2}s_{\alpha_1}s_{\alpha_4}s_{\alpha_3}$. The $W^u([2143])$ is then the subgroup generated by $\{s_{\alpha_2}, s_{\alpha_4}, s_{\alpha_5}\}$ and is diffeomorphic to $\mathbb{R}^3$. In terms of *handle body*, the critical point [2143] is identified as the product $D^2 \times D^3$ where D^n is the n-dimensional disc.

With this identification, we have the Morse decomposition of the manifold $\hat{H}_{\mathbb{R}}$,

$$\hat{H}_{\mathbb{R}} = \bigcup_{a\in W} W^u(a). \tag{4.2}$$

The index of the critical point $a \in W$ is defined as

$$\mathrm{Ind}(a) := \dim W^u(a) = \left|\Pi^u_a\right| \tag{4.3}$$

which is also given by the number of $*$s in the labelled Dynkin diagram, e.g. for $\mathfrak{sl}(6 : \mathbb{R})$, $\mathrm{Ind}([2143]) = 3$.

The Toda flow defines a (directed) graph which provides one-dimensional connections among the critical points corresponding to the one-dimensional flow of a $\mathfrak{sl}(2, \mathbb{R})$ subsystem. We call the graph the *Toda graph* and it is defined by

Definition 4.1 (Toda graph). *A directed graph is called a Toda graph if each vertex defined by* $\langle a\rangle = a^{-1}\langle e\rangle$ *with* $a \in W$ *has the connections to another vertex* $\langle b_i\rangle$ *by*

$$b_i = a s_{\alpha_i} \qquad \text{for} \quad i = 1, \ldots, l.$$

The direction in the connection between two vertices a and b is defined by

$$a \to b \qquad \text{if} \quad \ell(a) < \ell(b).$$

In order to construct a Morse complex, a vector field on the manifold must satisfy the *Morse–Smale condition*, that is, the intersection between (the manifolds) $W^u(a)$ and $W^s(b)$ for the critical points a and b must be transversal. However, the corresponding intersections in the case of the Toda lattice are, in general, *not* transversal. We have:

Definition 4.2 (Transversal connection (algebraic version)). *A connection* $a \to b$ *is transversal if*

(a) $\left|\Pi_a^u \cap \Pi_b^s\right| = \mathrm{Ind}(a) - \mathrm{Ind}(b)$,
(b) $\langle W^u(a), W^s(b)\rangle = W$,
(c) $\left|aW^u(a) \cap bW^s(b)\right| = \left|W_{\Pi_a^u \cap \Pi_b^s}\right|$.

This definition is motivated by:

Theorem 4.1. *For the Toda lattice vector field, each* closure, $\overline{W^u}(a)$ $(\overline{W^s}(a))$ $a \in W$ *of the unstable (stable, respectively) manifold is smooth. Moreover, each smooth manifold* $\overline{W^u}(a)$ $(\overline{W^s}(a))$ *is orientable and produces a cycle if and only if the subgroup* $W^u(a)$ $(W^s(a))$ *is Abelian. A connection* $a \to b$ *is transversal (definition (4.2)) if and only if* $\overline{W^u}(a)$, $\overline{W^s}(b)$ *intersect transversally. The intersection is diffeomorphic to a circle.*

The proof of the theorem can be obtained from the methods developed in [2], and the detail will be given elsewhere.

We call a graph with vertices given by W and oriented edges $a \to b$ satisfying the (algebraic) transversality conditions above with $\mathrm{Ind}(a) = \mathrm{Ind}(b) + 1$, a *Morse–Smale graph*, if in addition, (a) there is a perturbation of the Toda lattice which is Morse–Smale and has the same set of critical points (W), (b) $a \to b$ only if the manifolds $W^u(a)$ and $W^s(b)$ for this new vector field intersect transversally. A Morse–Smale vector field can be obtained by a small smooth perturbation of the Toda lattice as in [11]. We have confirmed that conditions (a)–(c) in definition 4.2 are sufficient to determine uniquely a Morse–Smale graph in the cases of $\mathfrak{g} \cong A_l$ up to $l = 3$. However, this may not be true, in general.

We now define a boundary map on the chain $\mathcal{C}_*$ of the cells of unstable Weyl groups $W^u(a)$, i.e.

$$\mathcal{C}_* = \bigoplus_{k=0}^{l} \mathcal{C}_k \qquad \mathcal{C}_k = \sum_{\mathrm{Ind}(a)=k} \mathbb{Z}\langle a\rangle \tag{4.4}$$

where $\langle a\rangle$ is the cell corresponding to $W^u(a)$. The chain $\mathcal{C}_k$ is the set of all cells $\langle a\rangle$ with the labelled Dynkin diagram having k number of $*$s. The boundary map $\partial_k : \mathcal{C}_k \to \mathcal{C}_{k-1}$ is then defined by

$$\partial_k : \langle a\rangle \longmapsto \partial_k\langle a\rangle = \sum_{\mathrm{Ind}(b)=k-1} [a; b]\,\langle b\rangle \tag{4.5}$$

where all the connections $a \to b$ are edges in the Morse–Smale graph, and the *incidence* number $[a; b]$ is given by

$$[a; b] = \left(1 + (-1)^{\sigma[a;b]}\right)(-1)^{\ell(a^{-1}b)+i} \tag{4.6}$$

with

$$\sigma[a;b] = \left|\{j : \epsilon_j \to \epsilon_j' < 0,\ \alpha_j \in \Pi_a^u\}\right|$$

where the index i is given by $\{s_{\alpha_i}\} = \Pi_a^u \cap \Pi_b^s$, i.e. the Dynkin diagram corresponding to $\langle b\rangle$ has 0 in the ith place in addition to the 0s in $\langle a\rangle$. The sign change $\epsilon_i \to \epsilon_i'$ under the connection $a \to b$ is defined as follows:

$$a^{-1}b \cdot (\epsilon_1, \ldots, \epsilon_l) = (\epsilon_1', \ldots, \epsilon_l') \tag{4.7}$$

where the initial signs ϵ_is are taken as

$$\epsilon_j = \begin{cases} + & \text{if} \quad \alpha_j \in \Pi_a^s \cup \Pi_b^u \\ - & \text{if} \quad \alpha_j \in \Pi_a^u \cap \Pi_b^s \end{cases}$$

and ϵ_j' is defined as (2.6). With this definition, we can determine the change of orientations of the hypersurfaces for $b_i > 0$ and $b_i < 0$ parallel to the surface given by $b_i = 0$ under the action of $x = a^{-1}b$.

4.2. Example of $A_l = \mathfrak{sl}(l+1;\mathbb{R})$

Let us first introduce the following elements of $W = S_{l+1}$, the symmetry group of the order of $l+1$:

$$s_{ij} := s_{\alpha_i} \cdots s_{\alpha_j} = [i \cdots j] \tag{4.8}$$

where the numbers $i \cdots j$ denote the consecutive numbers between i and j for $1 \leqslant i, j \leqslant l$. Then, for example, some (but not all) of the connections from the top cell $\langle e\rangle = (* \cdots *)$ to the cells labelled $(* \cdots * 0 * \cdots *)$ with 0 in the jth place are expressed by $s_{ij} : \langle e\rangle \to \langle s_{ij}\rangle$ with

$$\langle s_{ij}\rangle := s_{ij}^{-1} \cdot \langle e\rangle = s_{ji} \cdot \langle e\rangle \qquad \text{for} \quad 1 \leqslant i, j \leqslant l. \tag{4.9}$$

Note that the cell $\langle s_{ij}\rangle$ is isomorphic to $A_{j-1} \times A_{l-j}$ as an unstable manifold $W^u(s_{ij})$ with index $l-1$ generated by the Toda flows. In particular, all the cells of A_{l-1}-type are given by $\langle s_{i1}\rangle$ and $\langle s_{il}\rangle$ for $i = 1, \ldots, l$. We call the k-cells of A_k-type the *principal* part (of the k-cells), and those of the $A_{j_1} \times \cdots \times A_{j_n}$ with $j_1 + \cdots + j_n = k, n > 1$ the *whisker* part. The set of all the (principal) k-cells of A_k-type is denoted as $\mathcal{A}_k$. For example, the boundary of a principal k-cell labelled by $(0 \cdots 0 * \cdots * 0 \cdots 0)$ with $l-k$ zeros is written in the sum of the principal and the whisker parts of $(k-1)$-cells. We then define a boundary map $\mathring{\partial}$ on the principal k-cells into the projection of the boundary map ∂ on the principal parts of $(k-1)$-cells:

$$\mathring{\partial}_k : \mathcal{A}_k \longrightarrow \mathcal{A}_{k-1}. \tag{4.10}$$

For the boundary of the top cell, we have

$$\mathring{\partial}_l \langle e\rangle = \sum_{i=1}^{l} \big([e; s_{i1}]\langle s_{i1}\rangle + [e; s_{il}]\langle s_{il}\rangle\big) \tag{4.11}$$

where the incidence numbers are computed as

$$[e; s_{i1}] = [e; s_{l-i+1,l}] = 2(-1)^{i+1}(1 - \delta_{il}).$$

Thus the principal part of the boundary of the top cell consists of $2(l-1)$ cells of A_{l-1}-type, and the cells $\langle s_{1l}\rangle$ and $\langle s_{l1}\rangle$ are *not* in the part of the boundary. We also note that $\langle s_{1l}\rangle$ and $\langle s_{l1}\rangle$ are only cells of A_{l-1} type separated from the others and invariant under the subgroup generated by $W^u(s_{1l})$ for $\langle s_{1l}\rangle$ and $W^u(s_{l1})$ for $\langle s_{l1}\rangle$. Then one can identify the cells which are not included in any parts of the boundaries of A_k type as in the following proposition:

Proposition 4.1. *All the cells which are free from the boundaries of cells with higher indices are generated by the following commutative diagram starting from* $\langle a_{0,0}\rangle := \langle e\rangle$*:*

$$\begin{tikzcd} \langle a_{i,j}\rangle \arrow[r, "s_{l-j,i+1}"] \arrow[d, "s_{i+1,l-j}"'] & \langle a_{i,j+1}\rangle \arrow[d, "s_{i+1,l-j-1}"] \\ \langle a_{i+1,j}\rangle \arrow[r, "s_{l-j,i+2}"] & \langle a_{i+1,j+1}\rangle \end{tikzcd} \tag{4.12}$$

where $\langle a_{i,j}\rangle$ *represents a unique cell labelled with* $(\overbrace{0\cdots 0}^{i} * \cdots * \overbrace{0\cdots 0}^{j}) \in \mathcal{A}_{l-(i+j)}$.

Proof. The braid relation $[i\cdot i+1\cdot i] = [i+1\cdot i\cdot i+1]$ shows the commutativity of the diagram. One can also show in a similar way as in (4.11) that the principal part of the boundary of the $\langle a_{i,j}\rangle$ consists of $2(l-(i+j)-1)$ cells and is given by

$$\mathring{\partial}_{l-(i+j)}\langle a_{i,j}\rangle = 2\sum_{k=1}^{l-(i+j)}(-1)^{k+1}\left(1-\delta_{k,l-(i+j)}\right)\left(\langle b_{i+1,i+k}\rangle + \langle b_{l-j,l-j-k+1}\rangle\right) \tag{4.13}$$

where $\langle b_{i',j'}\rangle = s_{i',j'}\cdot\langle a_{i,j}\rangle$, and the cells $\langle b_{i+1,l-j}\rangle = \langle a_{i+1,j}\rangle$ and $\langle b_{l-j,i+1}\rangle = \langle a_{i,j+1}\rangle$ do not appear. □

The cells defined in proposition 4.1 give the *seed* elements of the 'principal graph' defined as the graph on the sets $\mathcal{A}_* := \bigoplus_{k=1}^{l}\mathcal{A}_k$ where the connections indicate the non-zero incidence numbers. Thus in the principal graph there are $\frac{l(l+1)}{2}$ disconnected subgraphs, each of which has a cell $\langle a_{i,j}\rangle$ as the highest-dimensional cell (the seed cell) in the subgraph with $\dim\langle a_{i,j}\rangle = l-(i+j)$. Then one can show that the pair $(\mathcal{A}_*, \mathring{\partial}_*)$ forms a subchain complex, that is, the boundary map satisfies $\mathring{\partial}_k \circ \mathring{\partial}_{k+1} = 0$.

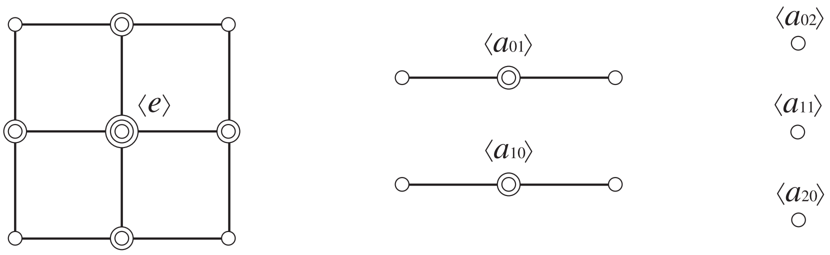

Figure 3. The principal graph for $A_3 = \mathfrak{sl}(4,\mathbb{R})$.

Figure 3 illustrates the example of $A_3 = \mathfrak{sl}(4;\mathbb{R})$. As in figure 3, one can identify the cells in $\mathcal{A}_k$ as $(k-1)$-dimensional cells in the graph consisting of $l(l+1)/2$ number of disconnected *hypercubes*. In each hypercube of dimension $k-1$, the top cell is represented by $\langle a_{i,j}\rangle$ with $k = l-(i+j)$, and the vertices represent A_1-cycles. For the case of A_3, we identify the seed cell $\langle e\rangle$ as the face (square), the cells in $\mathcal{A}_2$ as the edges and those in $\mathcal{A}_1$ as the vertices. Then counting the numbers of those cells, we obtain

Theorem 4.2. *The generating function (Poincaré polynomial)* $P(\mathcal{A}_*;q)$ *of the number of cells* $|\mathcal{A}_k|$ *is given by*

$$P(\mathcal{A}_*;q) = \sum_{k=1}^{l}|\mathcal{A}_k|q^{k-1} = \sum_{n=1}^{l} n(q+2)^{l-n}. \tag{4.14}$$

Proof. It is easy to see that the number of k-dimensional cells a_k in the n-dimensional hypercube is given by $\binom{n}{k}2^{n-k}$ so that we have

$$\sum_{k=0}^{n} |a_k| q^k = (q+2)^n.$$

From proposition 4.1, we have a number n of $(l-n)$-dimensional hypercubes in the principal graph. This asserts the theorem. □

As a corollary of theorem 4.2, we obtain

Corollary 4.1. *The Betti number of $H_1(\hat{H}_{\mathbb{R}}, \mathbb{Z})$ is given by*

$$b_1(\hat{H}_{\mathbb{R}}) := \mathrm{rank}(H_1) = P(\mathcal{A}_*; -1) = \frac{l(l+1)}{2}. \tag{4.15}$$

Proof. The total number of A_1-cycles is given by the number of vertices in the graph, i.e.

$$|Z_1| = \sum_{n=0}^{l-1} 2^n (l-n) = 2^{l+1} - (l+2).$$

From the graph, we can also find the number of boundaries, that is, in each graph of n-dimensional hypercubes there are $2^n - 1$ boundaries, and we have $l - n$ disconnected graphs in this dimension. Then we have

$$|B_1| = \sum_{n=1}^{l-1} (l-n) \times (2^n - 1)$$

and obtain the Betti number $b_1 = |Z_1| - |B_1|$ as stated. □

Although we have a complete characterization of the cells in terms of coloured Dynkin diagrams (section 3), we have not obtained explicitly a higher homology. It is, however, natural to consider the following conjecture on the Betti numbers b_k as the alternative sums of the numbers of *whiskers*:

$$b_k = \begin{cases} \displaystyle\sum_{n=k}^{l-k+1} \left|\mathcal{A}_n^{(k)}\right| (-1)^{n-k} & \text{for} \quad 1 \leqslant k \leqslant \dfrac{l+1}{2} \\ 0 & \text{for} \quad k > \dfrac{l+1}{2} \end{cases} \tag{4.16}$$

where $|\mathcal{A}_n^{(k)}|$ is the number of whiskers defined by

$$\left|\mathcal{A}_n^{(k)}\right| := \sum_{\substack{n_1+\cdots+n_k=n \\ 1\leqslant n_1 \leqslant\cdots\leqslant n_k}} \left|\mathcal{A}_{n_1} \times \cdots \times \mathcal{A}_{n_k}\right|.$$

Note here that all the k-cycles are given by the products of A_1-cycles, i.e. $|Z_k| = |\mathcal{A}_k^{(k)}|$. The conjecture is confirmed for the cases of $\mathfrak{g} \cong A_l$ up to $l = 3$.

5. Final remark

In this paper, we have studied the topology of the isospectral manifolds associated with the compactified level variety of the generalized Toda (Kostant–Toda) lattices on real split semisimple Lie algebras. The details of the decomposition based on the coloured Dynkin diagrams can be found in our recent paper [2], and the proofs of the results stated in sections 2 and 3 can also be found therein.

As a final remark, we would like to mention a possible extension of the present study for the full Kostant–Toda lattices which are recently shown to be integrable in [6]. Our methods may then shed some light on the structure of the real full flag manifold.

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